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Analysis
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Eigenschaften von Sinus, Cosinus, Sinus Hyperbolicus und Cosinus Hyperbolicus

Beweis von Satz 3.50

(i)

$$\cos(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 0^{2n} = \frac{(-1)^0}{0!} 0^0 = 1$$

$$\sin(0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n+1} = 0$$

(ii)

$$\begin{aligned}
\cos(-z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (-z)^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n}}{(2n)!} z^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \\
&= \cos(z)
\end{aligned}$$

$$\begin{aligned}
\sin(-z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (-z)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)(-1)^{2n}}{(2n+1)!} z^{2n+1} \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\
&= -\sin(z)
\end{aligned}$$

Beweis von Satz 3.51

Es gilt $\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z)$. Damit folgt

$$\begin{aligned}
\frac{1}{2}\left(\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)\right) &= \frac{1}{2}\left(\cos(z) + \mathrm{i}\sin(z) + \cos(-z) + \mathrm{i}\sin(-z)\right) \\
&\overset{3.50\ (ii)}{=} \frac{1}{2}\left(\cos(z) + \mathrm{i}\sin(z) + \cos(z) - \mathrm{i}\sin(z)\right) \\
&= \frac{1}{2} \cdot 2\cos(z) = \cos(z)
\end{aligned}$$

und

$$\begin{aligned}
\frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\right) &= \frac{1}{2\mathrm{i}}\left(\cos(z) + \mathrm{i}\sin(z) - \cos(-z) - \mathrm{i}\sin(-z)\right) \\
&\overset{3.50\ (ii)}{=} \frac{1}{2\mathrm{i}}\left(\cos(z) + \mathrm{i}\sin(z) - \cos(z) + \mathrm{i}\sin(z)\right) \\
&= \frac{1}{2\mathrm{i}} \cdot 2\mathrm{i}\sin(z) = \sin(z)
\end{aligned}$$

Beweis von Satz 3.52

(i)

$$\begin{aligned}
\sin^2(z) + \cos^2(z) &= \left(\frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\right)\right)^2 + \left(\frac{1}{2}\left(\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)\right)\right)^2 \\
&= -\frac{1}{4}\left(\exp(\mathrm{i}2z) - 2 + \exp(-\mathrm{i}2z)\right) + \frac{1}{4}\left(\exp(\mathrm{i}2z) + 2 + \exp(-\mathrm{i}2z)\right) \\
&= \frac{2}{4} + \frac{2}{4} = 1
\end{aligned}$$

(ii)

$$\begin{aligned}
& \cos(z)\cos(w) - \sin(z)\sin(w) \\
=\ & \frac{1}{2}\left(\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)\right)\frac{1}{2}\left(\exp(\mathrm{i}w) + \exp(-\mathrm{i}w)\right) \\
& -\frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\right)\frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}w) - \exp(-\mathrm{i}w)\right) \\
=\ & \frac{1}{4}\left(\exp(\mathrm{i}(z+w)) + \exp(\mathrm{i}(z-w)) + \exp(\mathrm{i}(-z+w)) + \exp(-\mathrm{i}(z+w))\right) \\
& +\frac{1}{4}\left(\exp(\mathrm{i}(z+w)) - \exp(\mathrm{i}(z-w)) - \exp(\mathrm{i}(-z+w)) + \exp(-\mathrm{i}(z+w))\right) \\
=\ & \frac{1}{4}\left(2\exp(\mathrm{i}(z+w)) + 2\exp(-\mathrm{i}(z+w))\right) \\
=\ & \frac{1}{2}\left(\exp(\mathrm{i}(z+w)) + \exp(-\mathrm{i}(z+w))\right) \\
=\ & \cos(z+w)
\end{aligned}$$

$$
\begin{aligned}
& \sin(z)\cos(w) + \cos(z)\sin(w) \\
= \; & \frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\right)\frac{1}{2}\left(\exp(\mathrm{i}w) + \exp(-\mathrm{i}w)\right) \\
& -\frac{1}{2}\left(\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)\right)\frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}w) - \exp(-\mathrm{i}w)\right) \\
= \; & \frac{1}{4\mathrm{i}}\left(\exp(\mathrm{i}(z+w)) - \exp(\mathrm{i}(-z+w)) + \exp(\mathrm{i}(z-w)) - \exp(-\mathrm{i}(z+w))\right) \\
& \frac{1}{4\mathrm{i}}\left(\exp(\mathrm{i}(z+w)) + \exp(\mathrm{i}(-z+w)) - \exp(\mathrm{i}(z-w)) - \exp(-\mathrm{i}(z+w))\right) \\
= \; & \frac{1}{4\mathrm{i}}\left(2\exp(\mathrm{i}(z+w)) - 2\exp(-\mathrm{i}(z+w))\right) \\
= \; & \frac{1}{2\mathrm{i}}\left(\exp(\mathrm{i}(z+w)) - \exp(-\mathrm{i}(z+w))\right) \\
= \; & \sin(z+w)
\end{aligned}
$$

Beweis von Satz 3.55

(i) Es gilt

$$
\begin{aligned}
\sinh(z) &= \frac{1}{2}\left(\exp(z) - \exp(-z)\right) \\
\cosh(z) &= \frac{1}{2}\left(\exp(z) + \exp(-z)\right)
\end{aligned}
$$

Damit folgt

$$
\begin{aligned}
\sinh(z) + \cosh(z) &= \frac{1}{2}\left(\exp(z) - \exp(-z)\right) + \frac{1}{2}\left(\exp(z) + \exp(-z)\right) \\
&= \frac{1}{2}\left(\exp(z) - \exp(-z) + \exp(z) + \exp(-z)\right) \\
&= \frac{1}{2}\cdot 2\exp(z) = \exp(z)
\end{aligned}
$$

(ii)

$$
\begin{aligned}
& \cosh^2(z) - \sinh^2(z) \\
= \; & \left(\frac{1}{2}\left(\exp(z) + \exp(-z)\right)\right)^2 - \left(\frac{1}{2}\left(\exp(z) - \exp(-z)\right)\right)^2 \\
= \; & \frac{1}{4}\left(\exp^2(z) + 2\exp(z)\exp(-z) + \exp^2(-z)\right) - \frac{1}{4}\left(\exp^2(z) - 2\exp(z)\exp(-z) + \exp^2(-z)\right) \\
= \; & \frac{1}{4}\cdot\left(2\exp(z)\exp(-z) + 2\exp(z)\exp(-z)\right) \\
= \; & \exp(z)\exp(-z) = \exp(z-z) = \exp(0) = 1
\end{aligned}
$$

(iii) Wir führen nur den Beweis für +.

$$\begin{aligned}
& \cosh(z)\cosh(w) + \sinh(z)\sinh(w) \\
= & \frac{1}{2}(\exp(z) + \exp(-z))\,\frac{1}{2}(\exp(w) + \exp(-w)) \\
& + \frac{1}{2}(\exp(z) - \exp(-z))\,\frac{1}{2}(\exp(w) - \exp(-w)) \\
= & \frac{1}{4}(\exp(z+w) + \exp(-z+w) + \exp(z-w) + \exp(-z-w)) \\
& + \frac{1}{4}(\exp(z+w) - \exp(-z+w) - \exp(z-w) + \exp(-z-w)) \\
= & \frac{1}{4}(2\exp(z+w) + 2\exp(-(z+w))) \\
= & \frac{1}{2}(\exp(z+w) + \exp(-(z+w))) \\
= & \cosh(z+w)
\end{aligned}$$

(iv) Wir führen nur den Beweis für +.

$$\begin{aligned}
& \sinh(z)\cosh(w) + \cosh(z)\sinh(w) \\
= & \frac{1}{2}(\exp(z) - \exp(-z))\,\frac{1}{2}(\exp(w) + \exp(-w)) \\
& + \frac{1}{2}(\exp(z) + \exp(-z))\,\frac{1}{2}(\exp(w) - \exp(-w)) \\
= & \frac{1}{4}(\exp(z+w) - \exp(-z+w) + \exp(z-w) - \exp(-z-w)) \\
& + \frac{1}{4}(\exp(z+w) + \exp(-z+w) - \exp(z-w) - \exp(-z-w)) \\
= & \frac{1}{4}(2\exp(z+w) - 2\exp(-(z+w))) \\
= & \frac{1}{2}(\exp(z+w) - \exp(-(z+w))) \\
= & \sinh(z+w)
\end{aligned}$$

(v)

$$\begin{aligned}
\sin(\mathrm{i}y) &= \frac{1}{2\mathrm{i}}(\exp(\mathrm{i}\mathrm{i}y) - \exp(-\mathrm{i}\mathrm{i}y)) \\
&= \frac{1}{2\mathrm{i}}(\exp(-y) - \exp(y)) \\
&= -\frac{1}{2\mathrm{i}}(\exp(y) - \exp(-y)) \\
&= \frac{\mathrm{i}}{2}(\exp(y) - \exp(-y)) \\
&= \mathrm{i}\sinh(y)
\end{aligned}$$

(vi)

$$\begin{aligned}
\cos(\mathrm{i}y) &= \frac{1}{2}\left(\exp(\mathrm{ii}y) + \exp(-\mathrm{ii}y)\right) \\
&= \frac{1}{2}\left(\exp(-y) + \exp(y)\right) \\
&= \frac{1}{2}\left(\exp(y) + \exp(-y)\right) \\
&= \cosh(y)
\end{aligned}$$

(vii)

$$\begin{aligned}
\cos(x + \mathrm{i}y) &\overset{3.52}{=} \cos(x)\cos(\mathrm{i}y) - \sin(x)\sin(\mathrm{i}y) \\
&\overset{(v),(vi)}{=} \cos(x)\cosh(y) - \mathrm{i}\sin(x)\sinh(y)
\end{aligned}$$

(viii)

$$\begin{aligned}
\sin(x + \mathrm{i}y) &\overset{3.52}{=} \sin(x)\cos(\mathrm{i}y) + \cos(x)\sin(\mathrm{i}y) \\
&\overset{(v),(vi)}{=} \sin(x)\cosh(y) + \mathrm{i}\cos(x)\sinh(y)
\end{aligned}$$